

Mirror Symmetry And Quantum Geometry

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ABSTRACT. Recently, mirror symmetry is derived as T-duality applied to gauge systems that flow to non-linear sigma models. We present some of its applications to study quantum geometry involving D-branes. In particular, we show that one can employ D-branes wrapped on torus fibers to reproduce the mirror duality itself, realizing the program of Strominger-Yau-Zaslow in a slightly different context. Floer theory of intersecting Lagrangians plays an essential role.

To the memory of Sung-Kil Yang

1. Introduction

Mirror symmetry has played important roles in exploring quantum modification of geometry in string theory. Things started with the discovery of mirror pairs of Calabi-Yau manifolds [1] with a subsequent application [2] to superstring compactifications. It had an immediate impact on enumerative geometry and motivated various mathematical investigations including the formulation of Gromov-Witten invariants. Another breakthrough was made through the recognition of D-branes as indispensable elements in string theory [3], which was preceded by Konstantinov's homological mirror symmetry [4]. Studies and applications of mirror symmetry involving D-branes have enriched our understanding of quantum geometry.

In particular, Strominger-Yau-Zaslow (SYZ) proposed, using the transformation of D-branes under T-duality, that mirror symmetry of Calabi-Yau manifolds is nothing but dualization of special Lagrangian torus fibrations [5]. This provides a very geometric picture of mirror symmetry that inspired many physicists and mathematicians.

Recently, another progress is made via an exact analysis of quantum field theory on the worldsheet [6]. Mirror symmetry is derived as T-duality applied to gauge systems [7] that flow to non-linear sigma models. This, however, turns sigma models into Landau-Ginzburg (LG) models, where the LG potential for the dual fields is generated by the vortex-anti-vortex gas of the high energy gauge system.

What is the relation of this and SYZ, both of which use T-duality applied to torus fibers? Since SYZ employ D-branes wrapped on the torus fibers, it is a natural idea to do the same. In this talk, we present some applications of [6] to study the properties of D-branes. In particular, we show that the study of D-branes wrapped on torus fibers indeed reproduces the LG mirror of [6]. Study of Floer homology for intersecting Lagrangians [8] plays an important role. We will also present other aspects of mirror symmetry involving D-branes.

2. T-Duality and D-Branes

Let us consider a closed string moving in the circle of radius R , which is described by a periodic scalar field $X \equiv X + 2\pi R$ on the worldsheet. The space of states is decomposed into sectors labeled by two conserved charges — the momentum $l \in \mathbb{Z}$ associated with the translation symmetry $X \rightarrow X + \text{constant}$, and the winding number $m \in \mathbb{Z}$ which counts how many times the string winds around the circle. The ground state in each sector has energy $12[(l/R)^2 + (Rm)^2] - 112$, which is invariant under

$$R \longleftrightarrow 1/R,$$

and $l \leftrightarrow m$. In fact, the sigma model on the circle of radius R is equivalent to the sigma model on the circle of radius $1/R$. This is called *T-duality*. The exchange of momentum and winding number can be described as the relation between the corresponding currents

$$(2.1) \quad \partial_t X = \partial_\sigma \tilde{X}, \quad \partial_\sigma X = \partial_t \tilde{X},$$

where (t, σ) are the time and space coordinates on the worldsheet and $\tilde{X} \equiv \tilde{X} + 2\pi/R$ is the coordinate of the T-dual circle.

Let us now introduce an open string to this system. We need to specify the boundary condition on the scalar field X at the worldsheet boundary, say, at $\sigma = 0$. Neumann boundary condition $\partial_\sigma X|_{\sigma=0} = 0$ corresponds to the freely moving end point, while Dirichlet boundary condition $\partial_t X|_{\sigma=0} = 0$ fixes the end point. They describe open strings ending on *D-branes*: the former is for a D1-brane wrapped on the circle while the latter is for a D0-brane at a point of the circle. By the relation (2.1), we see that T-duality exchanges the Neumann and Dirichlet boundary conditions. Thus, T-duality maps a D1-brane wrapped on the circle to a D0-brane at a point of the T-dual circle. The open string end point is charged under $U(1)$ gauge field on the D-brane. The holonomy $a = \int_{S^1} A$ parametrizes the gauge field configuration on the D1-brane wrapped on the circle. Under T-duality, this parameter is mapped to the position of the D0-brane in the T-dual circle. Therefore, T-dual of S^1 can be identified as the dual circle $H^1(S^1, U(1))$. This story generalizes to the higher dimensional torus T^n : T-duality inverts the radii of the torus, mapping a Dn -brane wrapped on T^n to a D0-brane at a point of the T-dual torus \tilde{T}^n , where the map provides the identification $\tilde{T}^n \cong H^1(T^n, U(1))$.

The same story applies to *supersymmetric* theories on the worldsheet, which are obtained by including fermionic fields as the superpartner of the scalar fields. When the target space is a Kahler manifold, the system has $(2, 2)$ *supersymmetry* which is an extended symmetry with four supercharges $Q_+, \bar{Q}_+, Q_-, \bar{Q}_-$ (\pm shows the worldsheet chirality). The supercharges are complex but are related by hermitian conjugation $\bar{Q}_\pm = Q_\pm^\dagger$. The complex coordinates of the target space are annihilated by \bar{Q}_\pm -variation, but are mapped to the partner Dirac fermions under the Q_\pm -variation. The simplest example is the cylinder $\mathbb{C}^\times = \mathbb{R} \times S^1$ with the (flat) product metric parametrized by the radius R of the circle. T-duality applied to the S^1 yields another cylinder $\tilde{\mathbb{C}}^\times = \mathbb{R} \times \tilde{S}^1$ with radius $1/R$. This is actually an example of *mirror symmetry*: Q_- and \bar{Q}_- are exchanged under the equivalence. T-duality maps the various D-brane configurations: D2-brane wrapped on \mathbb{C}^\times is mapped to D1-brane extending in the \mathbb{R} -factor of $\tilde{\mathbb{C}}^\times$; D1-brane wrapped on S^1 at a point in the \mathbb{R} -direction is mapped to D0-brane at a point of $\tilde{\mathbb{C}}^\times$.

Let us next consider a more interesting target space — the two-sphere S^2 . It can be viewed as the circle fibration over a segment, and one may ask what happens if T-duality is applied fiber-wise. Since T-duality inverts the radius of the circle, larger circle is mapped to smaller circle and smaller circle is mapped to larger circle, and one may naïvely expect that the dual geometry is as in Fig. 1. Since the size

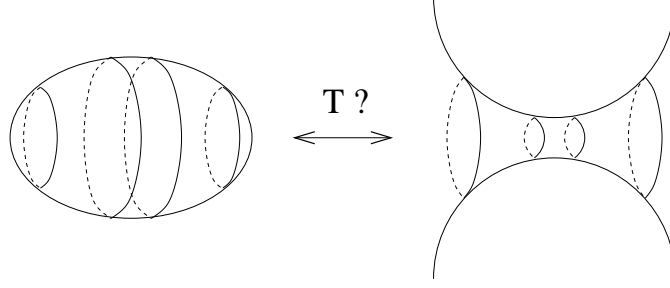


FIGURE 1. Is this what T-duality does?

of the dual circle blows-up toward the two ends, two holes effectively opens up and the dual geometry has topology of cylinder. This is consistent in one aspect: the conserved momentum associated with the $U(1)$ -isometry of S^2 (fiber-rotation) is mapped to the winding number of the dual system, which is conserved due to the cylindrical topology. However, another aspect is not clear. The winding number is not conserved in the original system because $\pi_1(S^2) = \{1\}$, and this should mean in the dual theory that the momentum is not conserved or the translation symmetry is broken. But how can it be broken? Is it because the metric is secretly not invariant under rotation of the cylinder?

What really happens under T-duality is as follows. It is true that the dual geometry has topology of cylinder. However, the dual theory is not just a sigma model but a model called *Landau-Ginzburg (LG) model*. It has a potential and Yukawa coupling terms determined by a holomorphic function of the target space, called the *superpotential*. Let us parametrize the dual cylinder by a complex coordinate Y which is periodic in the imaginary direction $Y \equiv Y + 2\pi i$. Then, the superpotential of the dual LG model is given by

$$(2.2) \quad W = e^{-Y} + e^{-t+Y}.$$

Here $t = r - i\theta$ is a complex parameter that corresponds to the data of the original S^2 sigma model: r is the area of the original S^2 and θ determines the B-field (it gives a phase factor $e^{ik\theta}$ to the path-integral measure for a worldsheet mapped to S^2 with degree k). It is this superpotential that breaks the translation symmetry $\text{Im}Y \rightarrow \text{Im}Y + \text{constant}$. This T-duality is a mirror symmetry, as in the example of cylinder.

This was derived in [6] by exact analysis of quantum field theory on the worldsheet. The derivation applies to the case where the target space is a general toric manifold X . A toric manifold, as S^2 is, can be viewed as the torus fibration over some base manifold, and T-duality sends the sigma model to a LG model. Suppose X is realized as the symplectic quotient of \mathbb{C}^N by $U(1)^k$ action $(z_i) \mapsto (e^{iQ_i^a \lambda_a} z_i)$ with the moment map equation $\sum_i Q_i^a |z_i|^2 = r^a$, and suppose the B-field is such that the path-integral weight is $e^{i \sum_a k_a \theta^a}$ for a map of multi-degree (k_a) . Then, the

dual geometry is an $(N - k)$ -dimensional cylinder defined by $\sum_i Q_i^a Y_i = r^a - i\theta^a$ for $Y_i \equiv Y_i + 2\pi i$, and the superpotential is given by $W = \sum_i e^{-Y_i}$. This mirror symmetry explains several observations made earlier [9–13]. The analysis of [6] also includes the derivation of the mirror pairs of Calabi-Yau hypersurfaces or complete intersections in toric manifolds [1, 14]. Furthermore, the method can be applied also to string backgrounds with non-trivial dilaton [15] and H-field [16].

We do not repeat the analysis of [6] in what follows. Instead, we will find some of the consequences of the mirror symmetry, especially on D-branes. We will see that D-brane analysis sheds new light on the duality itself. Recall that, for the circle sigma model, the dual circle was identified as the space of wrapped D1-branes, $\tilde{S}^1 \cong H^1(S^1, U(1))$. The same will happen here; the dual theory can be rediscovered by looking at the D-branes wrapped on the circle fibers of the toric manifold.

3. Supersymmetric D-branes

Abstractly, D-branes can be regarded as boundary conditions or boundary interactions on the worldsheet of an open string. We will focus on those preserving a half of the $(2, 2)$ worldsheet supersymmetry. There are two kinds of such D-branes [17]: A-branes preserving the combinations $Q_A = \overline{Q}_+ + Q_-$ and $Q_A^\dagger = Q_+ + \overline{Q}_-$; B-branes preserving $Q_B = \overline{Q}_+ + \overline{Q}_-$ and $Q_B^\dagger = Q_+ + Q_-$. Since mirror symmetry exchanges Q_- and \overline{Q}_- , A-branes and B-branes are exchanged under mirror symmetry.

Let us consider the sigma model on a Kähler manifold M . M can be considered as a complex manifold or as a symplectic manifold (with respect to the Kähler form ω). A D-brane wrapped on a cycle γ of M and supporting a unitary gauge field A is an A-brane if γ is a Lagrangian submanifold ($\omega|_\gamma = 0$) and A is flat ($F_A = 0$), while it is a B-brane if γ is a complex submanifold of M and A is holomorphic ($F_A^{(2,0)} = 0$). If we consider a LG model with superpotential W , there is a further condition that the W -image of γ is a straightline parallel to the real axis for A-branes and W is a constant on γ for B-branes [18, 19]. A-branes and B-branes are objects of interest from the point of view of symplectic geometry and complex analytic geometry, respectively. They are exchanged under mirror symmetry.

Of prime interest are the lowest energy states of open strings ending on D-branes. In particular, the supersymmetric ground states which correspond to massless open string modes. The theory of an open string stretched between two A-branes (or two B-branes) has one complex supercharge $Q = Q_A$ (or $Q = Q_B$). In many cases, it obeys the supersymmetry algebra

$$(3.1) \quad \begin{aligned} \{Q, Q^\dagger\} &= 2H, \\ Q^2 &= 0, \end{aligned}$$

where H is the Hamiltonian of the system. Then, the system can be regarded as the supersymmetric quantum mechanics (with infinitely many degrees of freedom) and the standard method [20] applies. In particular, there is a one-to-one correspondence between the ground states and the Q -cohomology classes. However, in some cases, the above algebra is modified and it can happen that

$$(3.2) \quad Q^2 \neq 0.$$

In such a case, the cohomological characterization of the ground states does not apply. (In fact, there is no supersymmetric ground states.) This does not happen for closed strings and is a new phenomenon peculiar to open strings.

In what follows, we study D-branes in the sigma model on S^2 or more general toric manifolds. In Sec. 4, we study A-branes in S^2 and the mirror B-branes in the LG model. We will see that we can reproduce the mirror duality through the study of D-branes. In Sec. 5, we study B-branes in S^2 and the mirror A-branes in LG.

4. Intersecting Lagrangians and their Mirrors

Let (M, ω) be a Kahler manifold. We study the open string stretched from one A-brane (γ_0, A_0) to another (γ_1, A_1) , where γ_i are Lagrangian submanifolds and A_i are flat $U(1)$ connections on them. Classical supersymmetric configurations are the ones mapped identically to the intersection points of γ_0 and γ_1 . However, quantum tunneling effects may lift the ground state degeneracy. Only the index $\text{Tr}(-1)^F = \#(\gamma_0 \cap \gamma_1)$ is protected from corrections.

To determine the actual ground state spectrum, one may apply Morse theory analysis of [20] to the space of open string configurations. The sigma model action defines a Morse function and its critical points are indeed the constant maps to the intersection points of γ_0 and γ_1 . Tunneling configurations are holomorphic maps from the strip to M such that the left and the right boundaries are mapped to γ_0 and γ_1 respectively, and the far past and the far future are asymptotic to the constant maps to $\gamma_0 \cap \gamma_1$. This usually leads to a cochain complex that models the original Q -complex. However, the ‘coboundary’ operator may fail to be nilpotent [8], $\partial^2 \neq 0$, which corresponds to $Q^2 \neq 0$.

As an example, consider two Lagrangian submanifolds in the complex plain $M = \mathbb{C}$ as depicted in Fig. 2. They intersect at two points q and p , and the

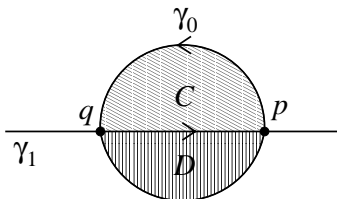


FIGURE 2. An example with $Q^2 \neq 0$

constant maps to them are the candidate supersymmetric configurations. The ‘cochain’ complex has \mathbb{Z}_2 grading that distinguishes p and q . There is one tunneling configuration from q to p — the holomorphic map from the strip to the region C . Then, the ‘coboundary’ operator acts as $\partial q = e^{-A(C)}p$ where $A(C)$ is the area of the region C . Likewise, we find $\partial p = e^{-A(D)}q$. Then, we see that

$$(4.1) \quad \partial^2 q = e^{-A(C)} e^{-A(D)} q = e^{-A(C \cup D)} q \neq 0.$$

The standard proof of $\partial^2 = 0$ does not apply here: there is a one parameter family of tunneling configurations from q to q , that starts with the composition of C and D at p and ends with the holomorphic disc $C \cup D$. This is a general phenomenon called “bubbling off of holomorphic discs”, which is peculiar to open string systems.

If ∂ happens to be nilpotent, one can define the cohomology group, which is known as the Floer cohomology group $HF((\gamma_0, A_0), (\gamma_1, A_1))$. This is the space of supersymmetric ground states of the open string system.

Let us next consider a LG model with superpotential W on a complex manifold Y . We study the open string stretched from a B-brane Z_0 to another Z_1 , where Z_i are complex submanifolds of Y on which W are constants. It is straightforward to show, using the canonical commutation relation, that

$$(4.2) \quad Q^2 = W|_{Z_1} - W|_{Z_0},$$

and there is no quantum correction to it. Thus, we see that $Q^2 \neq 0$ if the W -values of Z_0 and Z_1 do not agree. If they do agree, the space of supersymmetric ground states is the Q -cohomology group. There is a finite dimensional model of the Q -complex [21, 22]; It consists of anti-holomorphic forms on $Z_0 \cap Z_1$ with values in the exterior powers of $N_{Z_0} \cap N_{Z_1}$, on which the coboundary operator acts as $\bar{\partial} + \partial W \cdot$ (here N_{Z_i} is the normal bundle of Z_i in Y , and $\partial W \cdot$ is the contraction with the holomorphic 1-form ∂W). If Z_0 and Z_1 are points, the complex is non-trivial only if they are the same point, and the cohomology is the exterior power of the tangent space if the point is a critical point of W but vanishes if the point is not a critical point.

Eqn. (4.2), if non-zero, is the mirror counterpart of $Q^2 \neq 0$ for the intersecting Lagrangian systems. Note that we find (4.2) by purely classical analysis, in contrast to the case of A-branes where computation of Q^2 requires the analysis of quantum tunneling effect.

To make it explicit, let us come back to the mirror symmetry between the S^2 sigma model and the LG model with superpotential $W = e^{-Y} + e^{-t+Y}$. We consider A-branes wrapped on the S^1 fibers in the Fig. 1(Left). They are mapped under T-duality to B-branes at points on the Y -cylinder — their $\text{Re}(Y)$ and $\text{Im}(Y)$ coordinates are determined respectively by the location of the S^1 and the holonomy of the $U(1)$ gauge field. One can actually find a detailed map via a field theory analysis [21]. Let $Y = c - ia$ be the location of the mirror D0-brane. Then, the area of the disc bounded by the original D1-brane is c and the holonomy of the $U(1)$ connection is $e^{i(a-c\theta/r)}$. Let us now find the condition for $Q^2 = 0$ using the LG description. For two D0-branes at $Y = c_0 - ia_0$ and $Y = c_1 - ia_1$, we find

$$(4.3) \quad Q^2 = e^{-c_1+ia_1} + e^{-t+c_1-ia_1} - e^{-c_0+ia_0} - e^{-t+c_0-ia_0}.$$

There are two solutions to $Q^2 = 0$: $(c_1, a_1) = (c_0, a_0)$ and $(r - c_0, \theta - a_0)$. In the S^2 side, they correspond to two identical D1-branes (the same location and the same holonomy), and two D1-branes of opposite holonomies such that the inside area of one is equal to the outside area of the other. The Q -cohomology is non-trivial only if the two points are the same critical point of the superpotential W , which is $e^{-Y} = e^{-t/2}$ or $-e^{-t/2}$ in the present case. In such a case, the cohomology is the exterior power of the tangent space, $\wedge \mathbb{C} = \wedge^0 \mathbb{C} \oplus \wedge^1 \mathbb{C}$, which has one bosonic and one fermionic basis vectors. In the original S^2 sigma model, $e^{-c+ia} = \pm e^{-t/2}$ means that the inside and outside area of S^1 are the same and the holonomy is ± 1 . Thus, we find that mirror symmetry predicts

$$(4.4) \quad HF((\gamma_0, A_0), (\gamma_1, A_1)) = \begin{cases} \wedge \mathbb{C} & \text{if } \gamma_0 = \gamma_1 \text{ divides } S^2 \text{ into 12} \\ & \text{and } A_0 = A_1 \text{ has holonomy } \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

One can also directly compute the Floer homology group. Let us put the D1-branes in a position as in Fig. 3. For simplicity, we set off the θ -angle as well as the

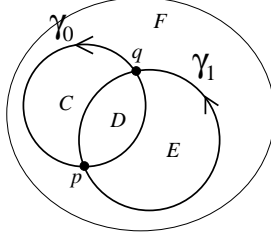


FIGURE 3. Intersecting Lagrangians in S^2

$U(1)$ connections on γ_i . The two circles intersect at two points q and p which represent ‘cochains’ of different degrees (the ‘complex’ is \mathbb{Z}_2 graded). The ‘coboundary’ operator acts as $\partial q = e^{-A(D)}p - e^{-A(F)}p$ and $\partial p = e^{-A(C)}q - e^{-A(E)}q$. Thus the square is $\partial^2 q = (e^{-A(C \cup D)} - e^{-A(D \cup E)} - e^{-A(C \cup F)} + e^{-A(E \cup F)})q$. If we denote the region inside γ_i by D_i , the area of the region outside γ_i is $r - A(D_i)$. We therefore find

$$(4.5) \quad \partial^2 q = \left[e^{-A(D_0)} - e^{-A(D_1)} - e^{-r+A(D_1)} + e^{-r+A(D_0)} \right] q$$

This vanishes if and only if $A(D_1) = A(D_0)$ or $A(D_1) = r - A(D_0)$, namely, when the inside area of γ_1 is equal to the inside or outside area of γ_0 . One can also show that the $U(1)$ holonomy of the two should be the same or opposite, respectively. This matches precisely with the LG result. In fact, (4.5) with holonomy included is identical to (4.3) under the map of variables mentioned before. Let us next compute the cohomology. Consider $A(D_0) = A(D_1)$ first. In such a case, $A(C) = A(E)$ and $A(F) - A(D) = A(F \cup C) - A(D \cup C) = r - A(D_1) - A(D_0) = r - 2A(D_0)$. Thus the coboundary operator acts as

$$\begin{aligned} \partial q &= e^{-A(D)}(1 - e^{-r+2A(D_0)})p \\ \partial p &= 0. \end{aligned}$$

The cohomology vanishes if $A(D_0) \neq r/2$, while it is non-vanishing if $A(D_0) = r/2$ — each of q and p generates the cohomology at its degree. Note that $A(D_0) = r/2$ is when γ_0 divides S^2 into half. One can also show that the cohomology is non-vanishing if the holonomy is ± 1 . We also find the same conclusion if we start with $A(D_1) = r - A(D_0)$. To summarize, we see that the result (4.4) of mirror symmetry is indeed correct.

We have seen a practical aspect of mirror symmetry in the study of D-branes. Computation of ground state spectrum of open string involves highly non-trivial analysis of quantum tunneling effect in the sigma model, while it is done by a simple classical manipulation in the mirror LG model. This story generalizes straightforwardly to more general toric manifolds.

There is another important aspect in the above analysis. We recall that, in the case of S^1 sigma model, it was enough to analyze the wrapped D1-brane to find the T-dual space; since D1-brane is T-dual to D0-brane, its moduli space is equivalent to the space of D0-branes of the T-dual theory, namely the dual space itself. In fact, the same applies here as well. By analysing the D1-branes wrapped on the S^1 -fibers, we find the cylinder as the dual space, but we also find the superpotential

(up to constant addition) through the computation of Q^2 , see (4.5). In other words, we can reproduce the mirror symmetry between toric sigma models and LG models by analyzing the D-branes.

This point of view shares its spirit with Strominger-Yau-Zaslow [5] who proposed, using D-branes, that mirror symmetry of Calabi-Yau manifolds is nothing but dualization of special Lagrangian fibrations. The latter has led, for example, to topological construction of mirror manifolds [23–25]. In attempts to make it more precise, the treatment of singular fibers constitutes the essential part where quantum corrections are expected to play an important role. (See e.g. [26] for a recent progress.) The example considered above includes singular fibers and we have shown how the quantum effect is taken into account. Although we have not dealt with *special* Lagrangian fibrations, we note that the above analysis applied to toric Calabi-Yau yields mirror manifolds consistent with SYZ program. For example, if we start with the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over \mathbb{CP}^1 , we obtain the LG model on $(\mathbb{C}^\times)^3$ with superpotential $W = e^{-Y_0}(e^{-Y_1} + e^{-Y_2} + e^{-t-Y_1-Y_2} + 1)$ as the mirror, which in turn is related [18] to the sigma model on the Calabi-Yau hypersurface $e^{-Y_1} + e^{-Y_2} + e^{-t-Y_1-Y_2} + 1 = uv$ in $\mathbb{C}^\times \times \mathbb{C}^\times \times \mathbb{C} \times \mathbb{C} = \{(e^{-Y_1}, e^{-Y_2}, u, v)\}$. This last mirror turns out to be consistent with SYZ topologically [27].

5. Holomorphic Bundles and their Mirrors

Let us consider an open string stretched between B-branes wrapped on M and supporting holomorphic vector bundles E_0 and E_1 . The zero mode sector of the open string Hilbert space is identified as the space $\Omega^{0,\bullet}(M, E_0^* \otimes E_1)$ where the supercharge Q acts as the Dolbeault operator. Thus, the space of supersymmetric ground states in the zero mode approximation is the Dolbeault cohomology or $\text{Ext}^\bullet(E_0, E_1)$. In particular, the index is

$$\text{Tr}(-1)^F = \chi(E_0, E_1).$$

In the full theory, some pairs states of neighboring R-charges could be lifted to non-supersymmetric states. The latter does not happen if $\text{Ext}^p(E_0, E_1)$ is non-zero only for even p (or odd p). An example of such a pair E_0, E_1 is from an *exceptional collection* [29], which is an ordered set of bundles $\{E_i\}$ where $\text{Ext}^p(E_i, E_i) = \delta_{p,0}\mathbb{C}$ while for $i < j$ $\text{Ext}^\bullet(E_j, E_i) = 0$ but $\text{Ext}^p(E_i, E_j)$ can be non-zero only for one value of p . For \mathbb{CP}^n , the set of line bundles $\{\mathcal{O}(i)\}_{i=j}^{j+n}$ is an exceptional collection ($\forall j \in \mathbb{Z}$).

Next we consider the LG model with superpotential W which has only non-degenerate critical points $\{p_i\}$. Gradient flows of $\text{Re } W$ originating from p_i sweep a Lagrangian submanifold γ_i whose W -image is a straight horizontal line emanating from the critical value $w_i = W(p_i)$. Thus, the D-brane wrapped on γ_i is an A-brane. Let us consider an open string stretched from γ_i to γ_j . Classical supersymmetric configurations are gradient flows of $-\text{Im } W$ from a point in γ_i to a point in γ_j . The index is the number of such gradient flows counted with an appropriate sign. It is

$$\text{Tr}(-1)^F = \#(\gamma_i^- \cap \gamma_j^+)$$

where γ_k^\pm is the deformation of γ_k so that the W -image is rotated at w_k with a small angle $\pm\epsilon$. Quantum mechanically, the paths of opposite signs are lifted by instanton effects and $|\#(\gamma_i^- \cap \gamma_j^+)|$ is in fact the number of supersymmetric ground states. One can also quantize the system using the Morse function determined by the LG action. This leads to the LG version of the Floer homology group $\text{HF}_W^\bullet(\gamma_i, \gamma_j)$.

(This is studied also by Y.-G. Oh [28].) If $\text{Im}w_i > \text{Im}w_j$ and there is no critical value between the W images of γ_i and γ_j , then $\#(\gamma_i^- \cap \gamma_j^+)$ is equal to the number S_{ij} of BPS solitons connecting p_i to p_j .

B-branes supporting holomorphic bundles on toric manifolds are mapped under mirror symmetry to A-branes in the LG models. Let us examine the detail in the S^2 sigma model and its mirror LG model $W = e^{-Y} + e^{-t+Y}$. The superpotential has two critical points p_{\pm} with critical values $w_{\pm} = \pm 2e^{-r/2+i\theta/2}$. We first set $\theta = 0$ so that w_{\pm} are on the real line. The simplest brane on S^2 is the $U(1)$ -bundle with trivial connection, or \mathcal{O} over \mathbb{CP}^1 . Since we are T-dualizing along the S^1 -fibers along which the holonomy is trivial, the dual has to be localized at a constant point in the dual fibers. It is a D1-brane at the horizontal line $\text{Im}Y = 0$ whose W -image is a straightline emanating from w_+ — the A-brane γ_+ . Another simple B-brane is the D0-brane at a point. Its dual is wrapped on S^1 but remains localized in the horizontal direction. The one at $\text{Re}Y = r/2$ has straight W -image and therefore is an A-brane, which we call γ_0 . Introducing n D0-branes to \mathcal{O} means turning on n

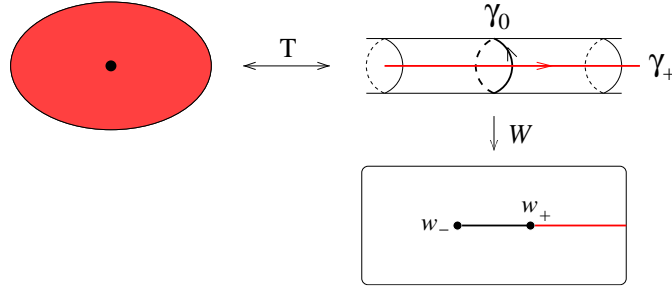


FIGURE 4

units of magnetic flux and produces the brane $\mathcal{O}(n)$. Its dual is the combination of γ_+ and $n\gamma_0$. For negative n , $n\gamma_0$ is understood to have the reversed orientation.

Let us now turn on a small positive θ . Then, w_+ is slightly above w_- in the imaginary direction, and γ_0 is no longer an A-brane. Instead we find a non-compact A-brane γ_- . See Fig. 5(Left). γ_- belongs to the homology class $\gamma_+ + \gamma_0$ and

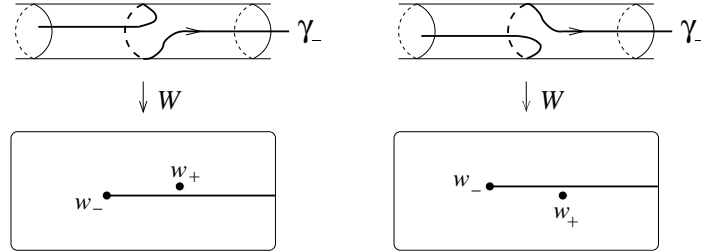


FIGURE 5

therefore is the mirror of $\mathcal{O}(1)$. We note that indeed $\chi(\mathcal{O}, \mathcal{O}(1)) = \#(\gamma_+^- \cap \gamma_-^+) = 2$. If we turn on a small negative θ , then w_+ is slightly below w_- , and we find the A-brane γ_- in the homology class $\gamma_+ - \gamma_0$ (Fig. 5(Right)). Since the orientation of γ_0 is reversed, γ_- is the mirror of $\mathcal{O}(-1)$ for this value of θ .

Note that γ_- for $\theta = +\epsilon$ and γ_- for $\theta = -\epsilon$ are related by Picard-Lefschetz formula $\gamma_-|_{\theta=+\epsilon} = (-\gamma_- + 2\gamma_+)|_{\theta=-\epsilon}$ where the coefficient 2 is $\#(\gamma_+^- \cap \gamma_-^+)$ at $\theta = +\epsilon$. On the other hand, the mirror bundles $\mathcal{O}(1)$ and $\mathcal{O}(-1)$ appear in the exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \text{Ext}^0(\mathcal{O}, \mathcal{O}(1)) \otimes \mathcal{O} \longrightarrow \mathcal{O}(1) \longrightarrow 0.$$

In such a case, $\mathcal{O}(-1)$ is said to be the *left mutation* of $\mathcal{O}(1)$ with respect to \mathcal{O} [29]. It was observed in [30, 31] (see also [32]) that mutation of exceptional bundles in certain Fano manifolds is related to Picard-Lefschetz monodromy that appears in the theory of BPS solitons in the sigma models. We can now understand it as a consequence of mirror symmetry in the case where the target space is a toric manifold. Related works has been done by P. Seidel [33].

6. Concluding Remarks

We have presented some applications of mirror symmetry between toric manifolds and LG models [6], especially to the study of D-branes. In particular, we have seen that D-branes wrapped on torus fibers can tell us about the mirror symmetry itself. Below, we comment on some materials that are not covered here.

Structure of integrable system in topological string theory, along with the matrix model representation, is an important aspect of quantum geometry. It was first discovered in topological gravity [34, 35], the case where the target space is a point. There are several observations suggesting that it may extend to more general target spaces. Here the mirror LG superpotential is expected to play an important role. For example, for \mathbb{CP}^1 model, $W = e^{-Y} + e^{-t+Y}$ is the Lax operator of the Toda lattice hierarchy [36, 12] under the replacement of Y by a differential operator (see also [37, 38]). For a projective space of higher dimension, the mirror superpotential also plays the role of Lax operator at least in genus zero [13]. Also, the Virasoro constraint [39] suggests existence of matrix model representations which in turn are related to integrable systems. Some beautiful story is waiting there to be discovered. Possible role of branes is interesting to explore.

Mirror symmetry has an application to enumerative geometry including holomorphic curves with boundaries. In the case of special Lagrangian submanifolds in Calabi-Yau three-folds, the number of holomorphic curves ending on the submanifolds enters into certain terms in the low energy effective action of the superstring theory. The number of holomorphic discs for a class of special Lagrangians in toric Calabi-Yaus are counted in [40] by computing the space-time superpotential terms in the mirror side. There are several related works including the mathematical tests [41, 42] of [40].

The derivation of [6] itself applies only to toric manifolds and submanifolds therein that are realized as vacuum manifolds of *abelian* gauge systems. There are some observations that suggest the form of LG-type mirror for Grassmann and flag manifolds [13, 43] which are realized as the vacuum manifolds of *non-abelian* gauge systems. It would be a challenging problem to find or derive the mirror of such manifolds and others. The method presented here using D-branes is possibly of some use. Another possible way is to consider compactification of three-dimensional mirror symmetry [44]. This works for the abelian gauge systems [45] and many examples of non-abelian mirror pairs are known in three-dimensional gauge theories (e.g. [46]).

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